On the Karhunen-Loève basis for continuous mechanical systems

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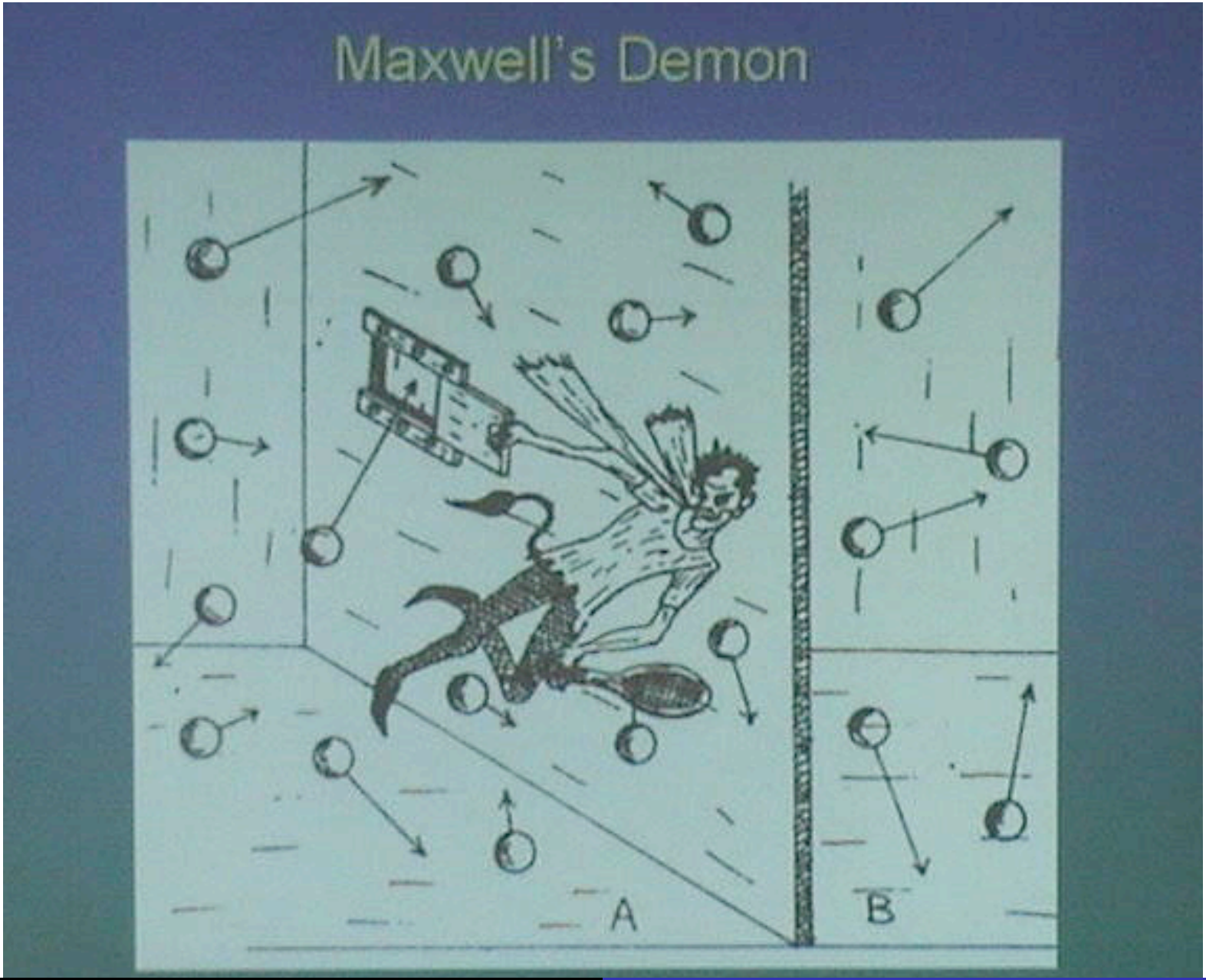
Some perspective to introduce the talk

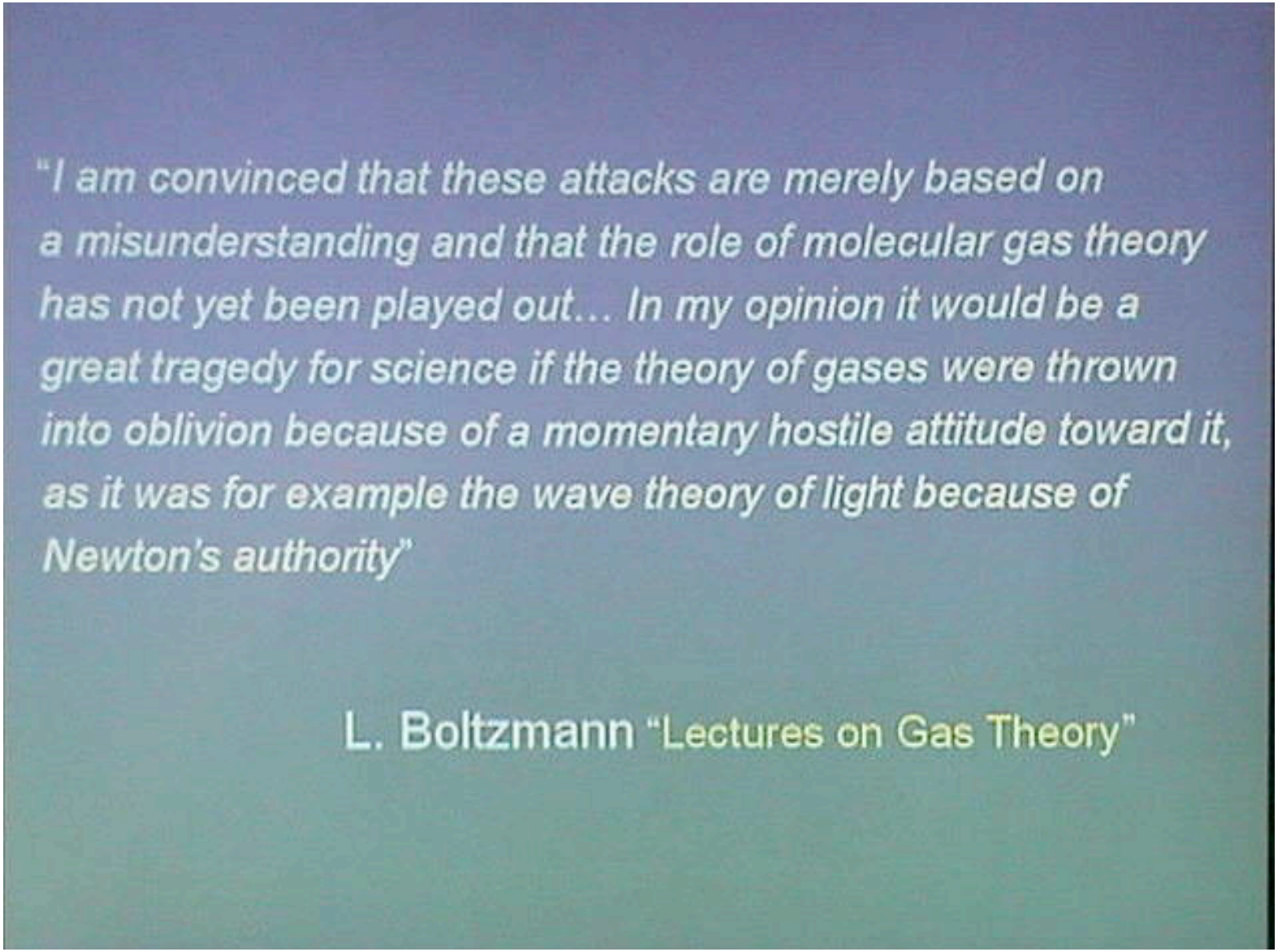
- Newton
- Maxwell
- Boltzmann
- Carnot: beginning of Thermodynamics
- A side comment

Newton

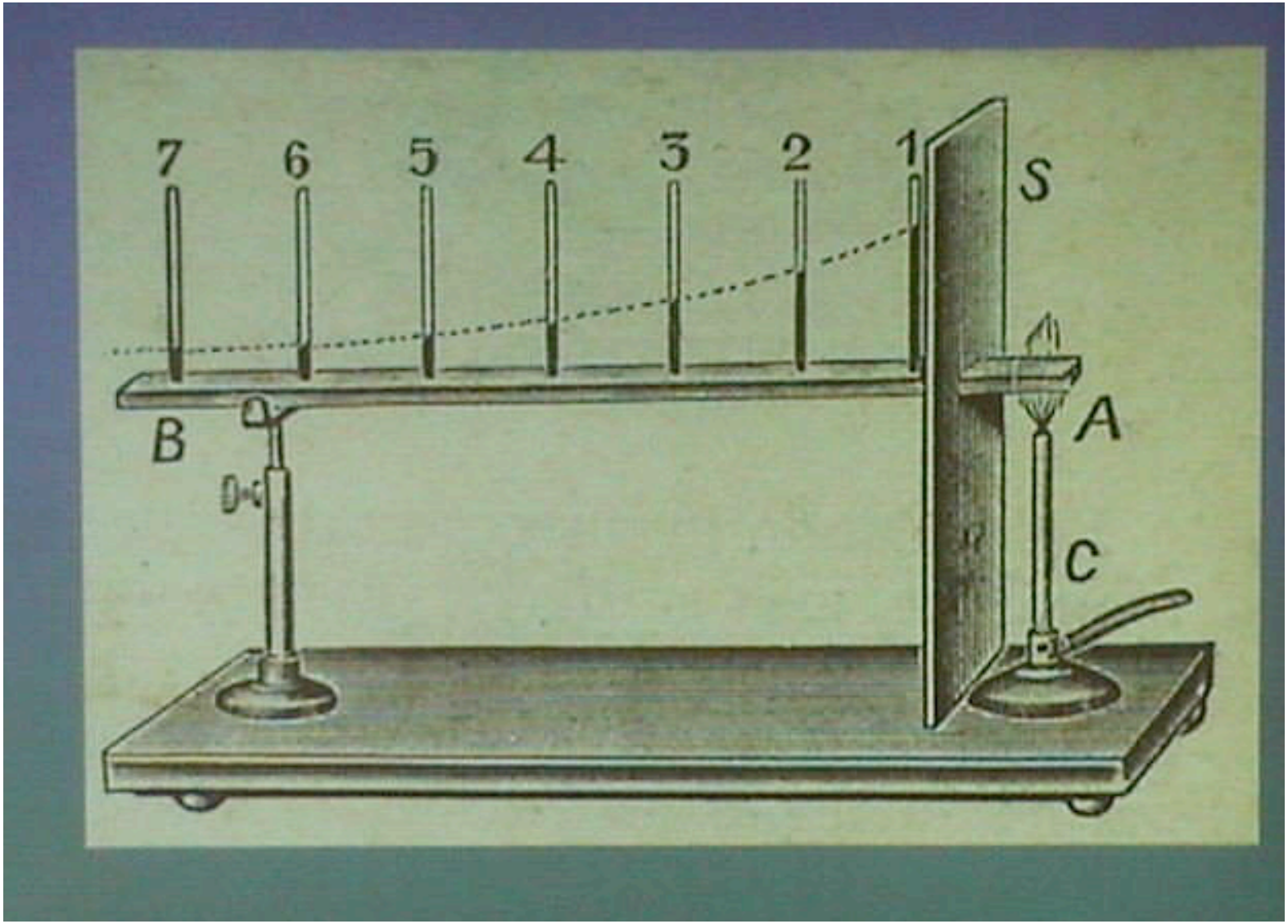
Maxwell



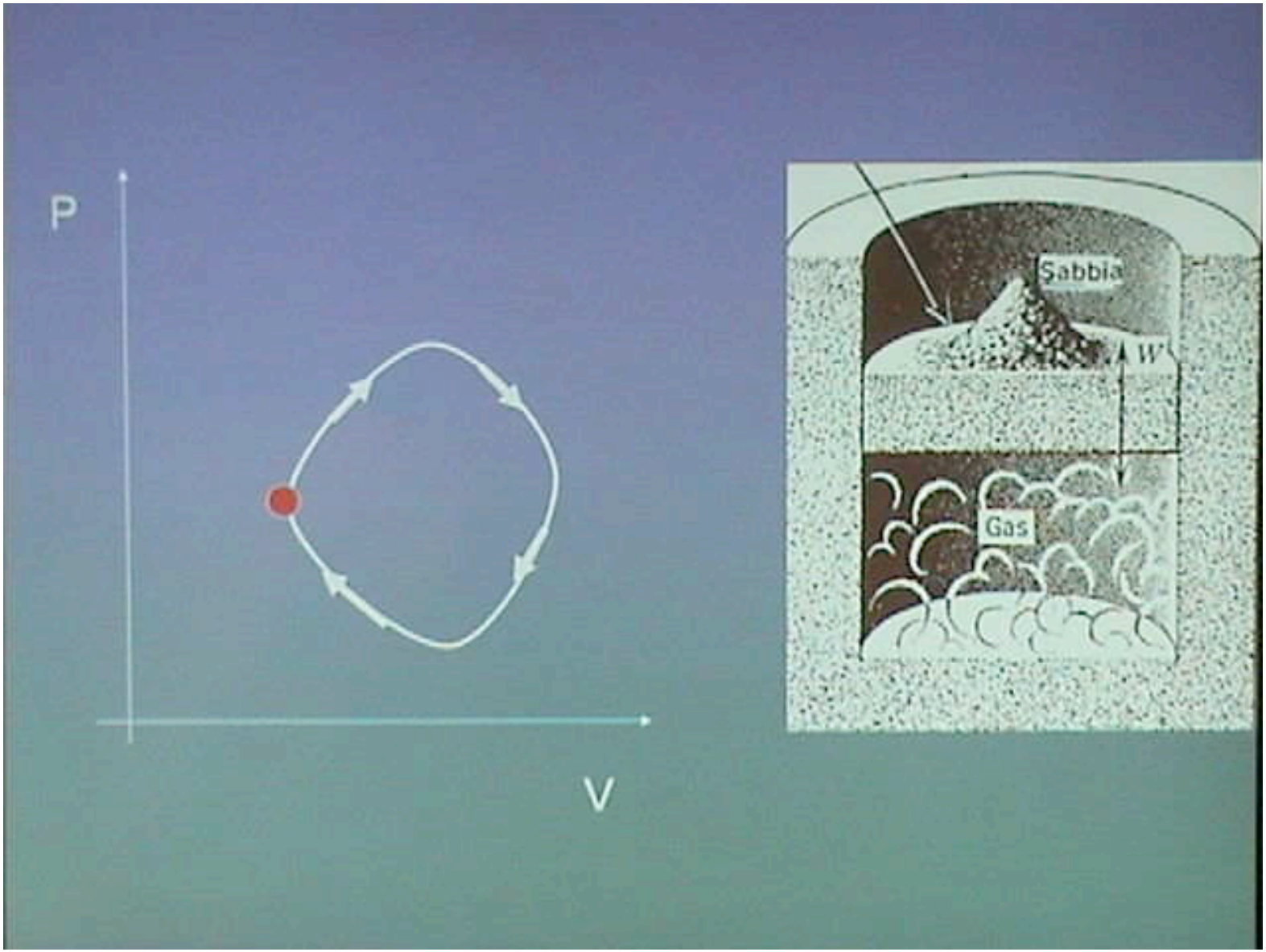
Boltzmann


"I am convinced that these attacks are merely based on a misunderstanding and that the role of molecular gas theory has not yet been played out... In my opinion it would be a great tragedy for science if the theory of gases were thrown into oblivion because of a momentary hostile attitude toward it, as it was for example the wave theory of light because of Newton's authority"
L. Boltzmann "Lectures on Gas Theory"

Beginning of Thermodynamics

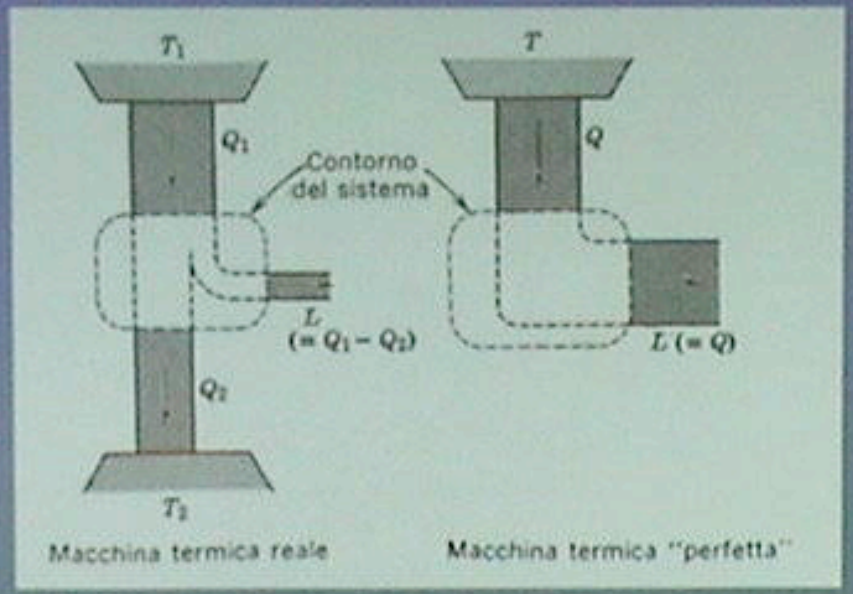


The idea of Carnot



Carnot's Theorem

Carnot Theorem



The efficiency of any thermal engine only depends on the temperatures of the two thermostates !!!

$$\eta = 1 - \frac{T_2}{T_1}$$

Why theorems are important

Let $\mathbb{N}$ be the set of natural numbers $1, 2, 3, ...$

Question: Find the greatest natural number.

Suppose N is the solution.
If $N \neq 1$ then $N^2 > N$, so $N \neq 1$ can not be the solution.
Then $N = 1$ is the solution

Answer: $N = 1$ is the greatest natural number.

Outline: Naive look/More detailed look

- Some history-main applications
- Main ideia of KL decomposition
- The mathematical problem
- Construction of the KL basis
- How to combine KL with Galerkin
- Reduced model given by KL
- Practical question: how to compute
- An example

- Different guises of KL; basic ingredients
- Karhunen-Loève expansion: main hypothesis
- Karhunen-Loève Theorem
- Basic properties
- Applications to Random Mechanics
- Examples

Some history

Beginning: works in Statistics and Probability and Spectral Theory in Hilbert Spaces. Some contributions:

- Kosambi (1943)
- Loève (1945)
- Karhunen (1946)
- Pougachev (1953)
- Obukhov (1954)

Applications:

- Lumley (1967): method applied to Turbulence
- Sirovich (1987): snapshot method

An important book appeared in 1996: Holmes, Lumley, Berkooz. In Solid Mechanics the applications started around 1993. In finite dimension it appears under different guises:

- Principal Component Analysis (PCA): Statistics and image processing
- Empirical orthogonal functions: Oceanography and Metereology

Main Applications

- Data analysis: Principal Component Analysis (PCA)
- Reduced models, through Galerkin approximations
- Dynamical Systems: to understand the dynamics
- Image processing
- Signal Analysis

Two main purposes:

- order reduction by projecting high-dimensional data in lower-dimensional space
- feature extration by revealing relevant but unexpected structure hidden in the data

Main idea of KL decomposition

In plain words Key idea of KL is to reduce a large number of interdependent variables to a much smaller number of uncorrelated variables while retaining as much as possible of the variation in the original data.

more precisely Suppose we have an ensemble $\{u_k\}$ of scalar fields, each being a function defined in $(a, b) \subset \mathbb{R}$. We work in a Hilbert space $L^2((a, b))$.
We want to find a (orthonormal) basis $\{\psi_n\}_{n=1}^{\infty}$ of L^2 that is optimal for the given data set in the sense that the finite dimensional representation of the form

$$\hat{u}(x) = \sum_{k=1}^{\infty} a_k \psi_k(x)$$

describes a typical member of the ensemble better than representations of the same dimension in any other basis.
The notion of typical implies the use of an average

The mathematical problem

Suppose, for simplicity, we have just one function ψ

$$max_{\psi \in L^2} \frac{E(|<u,\psi>|^2)}{\|\psi\|^2}$$

This implies

$$J(\psi) = E(|<u,\psi>|^2) - \lambda(\|\psi\|^2 - 1)$$

$$\frac{d}{ds} J(\psi + \varepsilon\phi)|_{\varepsilon=0} = 0$$

$$\int_a^b R(x,y)\psi(y)dy = \lambda\psi(x)$$

with $R(x,y) = E(u(x)u(y))$

Construction of the KL basis

- Construct R(x,y) from the data
- Solve the eigenvalue problem:

$$\int_{\mathscr{D}} R(x,y)\psi(y)dy = \lambda\psi(x)$$

  to get the pair (λ_i, ψ_i)
- If u is the field then the N-order approximation of it is

$$\hat{u}_N(t,x) = E(u(t,x)) + \Sigma_{i=1}^{N} a_i(t)\psi(x)$$

- To make predictions use the Galerkin method taking the ψ's as trial functions

Galerkin projections

Suppose we have a dynamical system governed by

$$\begin{array}{lcll} \frac{\partial v}{\partial t} & = & A(v) & v \in (a,b) \times \mathscr{D} \to \mathbb{R}^n \\ v(0,x) & = & v_0(x) & \textit{initial condition} \\ B(v) & = & 0 & \textit{boundary condition} \end{array}$$

The Galerkin method is a discretization scheme for PDE based on separation of variables.
One searches solutions in the form:

$$\hat{v}(x) = \sum_{k=1}^{\infty} a_k \psi_k(x)$$

Reduced equations

The reduced equation is obtained making the error of the approximation orthogonal to the first N KL elements of the basis.

$$\begin{aligned} errorequation(t,x) &= \frac{\partial \hat{v}}{\partial t} - A(\hat{v}) \\ errorinicond(x) &= \hat{v}(0,x) - v_0(x) \end{aligned}$$

$$< errors, \psi_i(x) >= 0 \ for \ i = 1, ..., N.$$

$$\begin{aligned} \frac{da_i}{dt}(t) &= \int_{\mathscr{D}} A(\Sigma_{n=1}^{N} a_n(t)\psi_n(x))\psi_i(x)dx \quad for \ i = 1, ..., N \\ a_i(0) &= \int_{\mathscr{D}} v_0(x)\psi_i(x)dx \quad for \ i = 1, ..., N \end{aligned}$$

Computation of the KL basis: Direct method

In this method, the displacements of a dynamical system are measured or calculated at N locations and labeled $u_1(t,x_1), u_2(t,x_2), \ldots, u_N(t,x_N)$. Sampling these displacements M times, we can form the following $M \times N$ ensemble matrix:

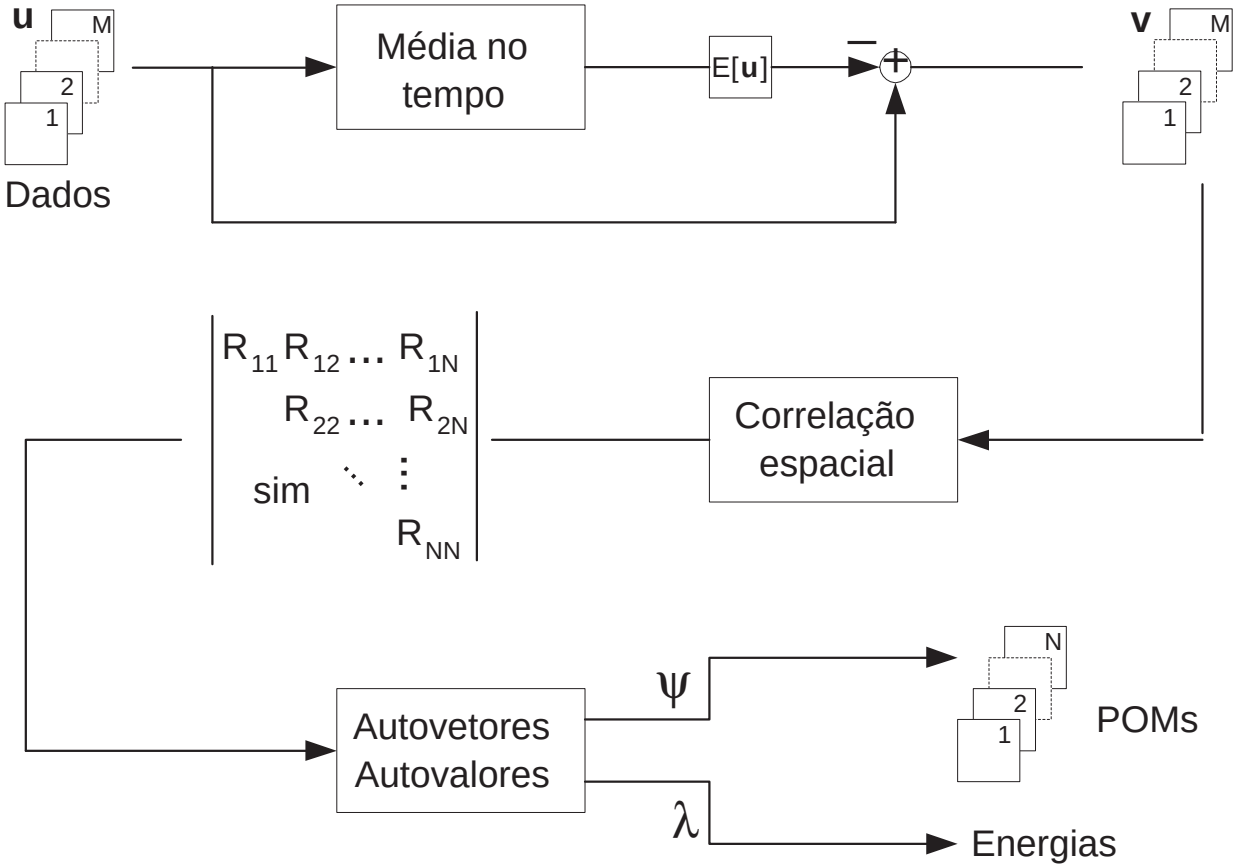
$$\mathbf{U} = \left[\begin{array}{cccc} \mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_N \end{array} \right] = \left[\begin{array}{cccc} u_1(t_1,x_1) & u_2(t_1,x_2) & \ldots & u_n(t_1,x_N) \\ \vdots & \vdots & \ddots & \vdots \\ u_1(t_M,x_1) & u_2(t_M,x_2) & \ldots & u_n(t_M,x_N) \end{array} \right]$$

Thus, the spatial correlation matrix of dimension $N \times N$ is formed as

$$\mathbf{R}_u = \frac{1}{M} \mathbf{U}^T \mathbf{U}.$$

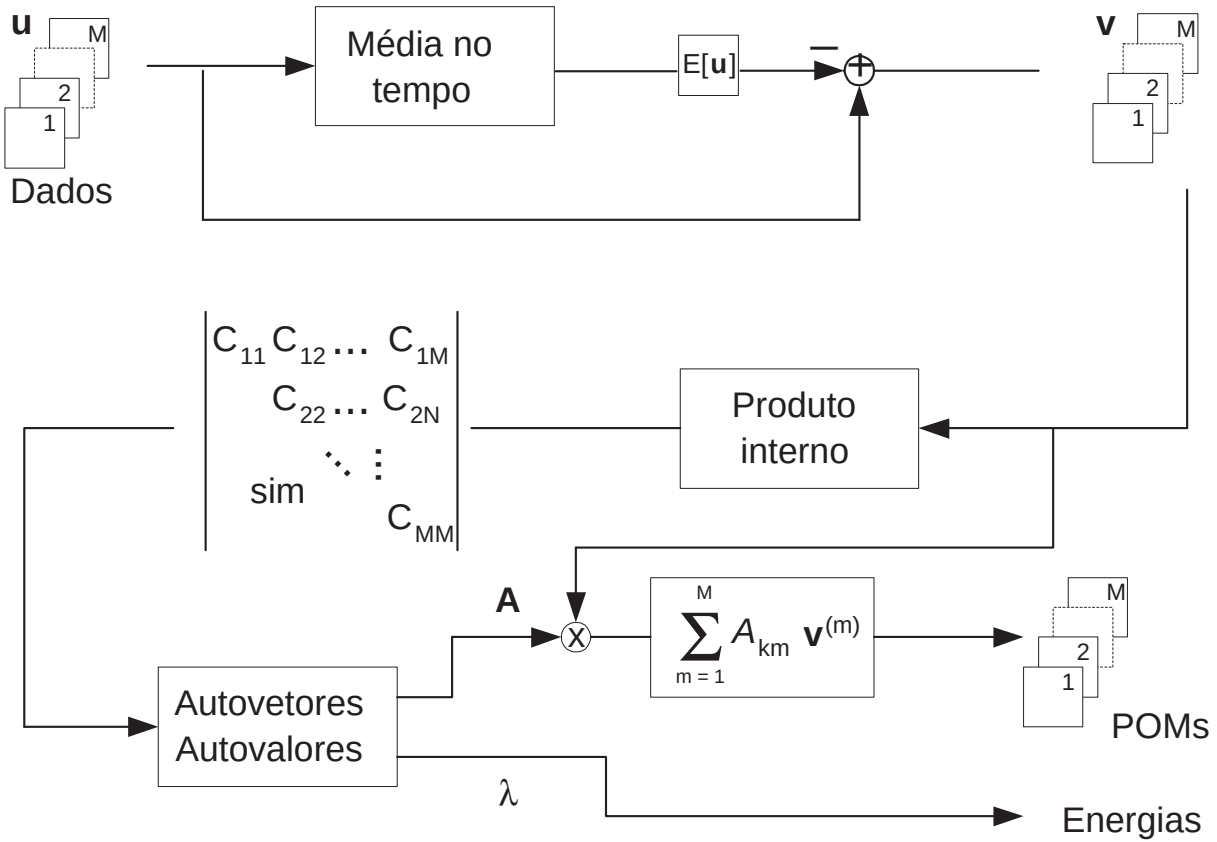
The PO modes are then given by the eigenvectors of $\mathbf{R}$,

Direct method



Algoritmo de implementação do método direto.

Snapshot method



Algoritmo de implementação do método dos retratos.

Different guises of KL; basic ingredients

Sets:

- $\mathscr{D} \subset \mathbb{R}^l$
- Ω space of events
- $\mathbb{R}^n$ codomain of functions

$L^2(\mathscr{D}, \mathbb{R}^n)$ is a Hilbert space of functions with inner product $<,>_{\mathscr{D}}$ and associated norm $\|.\|_{\mathscr{D}}$. The elements of this space are deterministic functions.
$(\Omega, \mathscr{F}, P)$ is a probability space, $\mathscr{F}$ is a sigma-algebra and P a probability measure. $\omega \in \Omega$ is an event, that is a realization of a random function.
The mean value of a random variable X is $E(|X|) = \int_\Omega X(z) dP(z)$ with

$$\begin{array}{rccc} X: & \Omega & \rightarrow & \mathbb{R}^n \\ & z & \mapsto & X(z) \end{array}$$

Ω also has a Hilbert space structure, noted $L^2(\Omega, \mathbb{R}^n)$, if we put the inner product $<,>_\Omega = E(|XY|)$ and the associated norm is $\|.\|_\Omega$

Basic ingredients of KL

In order to compute KL basis one needs two basic ingredients:

- a L_2 space of functions
- an averaging operator

In the literature we find mainly three main forms of KL decompositions. To understand their similarities and differences it is worth to think of the fields as defined in a cartesian product of two sets, that will provide the main ingredients we just mentioned

$$\begin{array}{rccc} X: & \mathscr{D} \times \Omega & \to & \mathbb{R}^n \\ & (z, \omega) & \mapsto & X(z, \omega) \end{array}$$

We have the following interpretation:

$X(z,.)$ is a random variable, that is, all possible realizations of a field for fixed $z \in \mathscr{D}$. We need the averaging operator to do statistics with this random variables, one for each $z \in \mathscr{D}$.

$X(.,\omega)$ this is a realization of a field, hence a function of $L_2(\mathscr{D}, \mathbb{R}^n)$. Physical quantities are defined in terms of

Karhunen-Loève expansion: main hypothesis

Let us consider a random field $\{X(z)\}_{z\in\mathscr{D}}$ defined on a probability space $(\Omega,\mathscr{F},P)$

$$\begin{array}{rccc} X: & \mathscr{D}(\subset\mathbb{R}^l)\times\Omega & \to & \mathbb{R}^n \\ & (z;\omega) & \mapsto & X(z;\omega) \end{array}$$

Assumption I: $\{X(z)\}_{z\in\mathscr{D}}$ is a second-order random field i.e.

$$E(\|X(z)\|^2) = E(< X(z), X(z) >) < \infty, \forall z \in \mathscr{D}$$

$E(.)$ denotes the ensemble average and ¡ , ¿ is the inner product in $\mathbb{R}^n$.

Assumption II: $\{X(z)\}_{z\in\mathscr{D}}$ is continuous in quadratic mean i.e.

$$\|X(z+h)-X(z)\|^2_{L^2(\Omega,\mathbb{R}^n)} \to 0 \text{ as } h \to 0.$$

Under Assumption **I** and **II**

- $\forall z \in \mathscr{D}$, $X(z) \in L^2(\Omega, \mathbb{R}^n)$ (with $< Y_1, Y_2 >_\Omega = E(< Y_1, Y_2 >)$).
- Second order moment characteristics:
  $m_X(z) = E(X(z))$
  $R_X(z_1, z_2) = E(X(z_1) \otimes X(z_2))$
  $C_X(z_1, z_2) = E((X(z_1) - E(X(1))) \otimes (X(z_2) - E(X(z_2))))$
- When the random field is mean zero valued, then $C_X = R_X$. We will assume in the sequel that $\{X(z)\}_{z \in \mathscr{D}}$ is a mean zero valued field.
- The correlation function C_X is continuous on $\mathscr{D} \times \mathscr{D}$.

The self-adjoint operator

According to our assumptions, the integral operator,

$$\begin{array}{rccl} Q: & L^2(\mathscr{D},\mathbb{R}^n) & \to & L^2(\mathscr{D},\mathbb{R}^n) \\ & \psi & \mapsto & (Q\psi)(z)=\displaystyle\int_{\mathscr{D}} C_X(z,z')\psi(z')dz', \end{array}$$

with kernel $C_X(z,z')$, defines a continuous self-adjoint Hilbert-Schmidt operator on the Hilbert space $L^2(\mathscr{D},\mathbb{R}^n)$.

Eigenvalues property:

The operator Q has a countable number of eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n \geq \cdots$,
i.e.

$$(Q\psi_n)(z) = \lambda_n \psi_n(z)$$

where

$$\psi_1, \cdots, \psi_n, \cdots,$$

denote the associated eigenfunctions.
The set of eigenfunctions constitutes a orthonormal basis of $L^2(\mathscr{D}, \mathbb{R}^n)$

$$< \psi_n, \psi_m >_{\mathscr{D}} = \int_{\mathscr{D}} < \psi_n(z), \psi_m(z) > dz = \delta_{nm}$$

where $<,>_{\mathscr{D}}$ denotes the inner product in $L^2(\mathscr{D}, \mathbb{R}^n)$ with the associated norm $\|.\|_{\mathscr{D}}$.

Karhunen-Loève Theorem

The Karhunen-Loève theorem states that a continuous second-order random field can be expanded in a series of the eigenfunctions, ψ_n, as

$$X(z) = \sum_{n=1}^{\infty} \xi_n \psi_n(z) \text{ (in } L^2(\Omega, \mathbb{R}^n))$$

where $\xi_1, \xi_2, \cdots, \xi_n, \cdots$ are scalar uncorrelated random variables defined by

$$\xi_n = \int_{\mathscr{D}} < X(z), \psi_m(z) > dz$$

with

$$E(\xi_n \xi_m) = \lambda_n \delta_{nm} = \begin{cases} \lambda_n & \text{if} \quad n = m \\ 0 & \text{if} \quad n \neq m \end{cases}$$

The $\{\psi_k\}$ are named the KL modes (also, Principal Orthogonal modes, POM).

Energy property

The eigenvalues, λ_n, of Q are related to the mean "energy" of the random field according to the following relation

$$E(\|X\|^2_{<,>_{\mathscr{D}}}) = \sum_{n=1}^{\infty} \lambda_n.$$

Optimality property

The Karhunen-Loève expansion satisfies the following optimality property:

$$E(\|X - \sum_{k=1}^{q} \xi_k \psi_k(z)\|_{\mathscr{D}}^2) \leq E(\|X - \sum_{k=1}^{q} \tilde{\xi}_k \tilde{\psi}_k(z)\|_{\mathscr{D}}^2)$$

for any integer q and any arbitrary orthogonal basis $(\tilde{\psi}_k)_{k\geq 1}$ of $L^2(\mathscr{D}, \mathbb{R}^n)$ where $\tilde{\xi}_1, \tilde{\xi}_2, \cdots, \tilde{\xi}_k, \cdots$ are scalar random variables given by

$$\tilde{\xi}_k = \int_{\mathscr{D}} < X(z), \psi_k(z) > dz$$

It is optimal in the sense that given a fixed number q of modes, no other linear decomposition can contain as much energy as the KL expansion.

Summary: Karhunen-Loève Theorem

$\{X(z)\}_{z\in\mathscr{D}}$ defined on a probability space $(\Omega,\mathscr{F},P)$
Covariance matrix function $C_X(z_1,z_2)$

$$X(z;\omega)=\sum_{n=1}^{\infty}\xi_n(\omega)\psi_n(z) \text{ in } L^2(\Omega,\mathbb{R}^n)$$

with

$$\begin{cases} \psi_n: & \text{eigenfunctions } (Q\psi)(z)=\int_{\mathscr{D}} C_X(z,z')\psi(z')dz' \\ & \int_{\mathscr{D}} <\psi_n(z)\psi_m(z)>dz=\begin{cases} 1 & \text{if} \quad n=m \\ 0 & \text{if} \quad n\neq m \end{cases} \text{ in } L^2(\mathscr{D},\mathbb{R}^n) \\ \xi_n: & \text{scalar random variables } \xi_n(\omega)=\int_{\mathscr{D}} <X(z,\omega),\psi_m(z)>dz \\ & E(\xi_n\xi_m)=\begin{cases} \lambda_n & \text{if} \quad n=m \\ 0 & \text{if} \quad n\neq m \end{cases} \end{cases}$$

Applications to Random Mechanics

In random mechanics, the random characteristics have often modeled using random fields $\{u(z)\}_{z\in\mathscr{D}}$ where the domain $\mathscr{D}$ is

either	$\mathscr{D}=\mathscr{D}_x\subset\mathbb{R}^p$(with $p=1,2$, or 3)	static problems
or	$\mathscr{D}=\mathscr{D}_t\times\mathscr{D}_x\subset\mathbb{R}\times\mathbb{R}^p$	dynamics problems

Without loss of generality, we assume $\mathscr{D}_t=[0,T]$ where $T\in\mathbb{R}+$. In order to find a flow model that still reveals the main features contained in the dynamics, one often searches for an expansion in the variables separated form

$$u(t,x)=\sum_{k=1}^{\infty}a_k(t)\phi_k(x)$$

where ϕ_k are deterministic $\mathbb{R}^n$-valued functions, and $\{a_k(t)\}_{t\in\mathscr{D}_t}$ are scalar time random processes.
Let us see how to adapt the KL theory to these cases.

Approach 1

Apply the KL theorem to random field $\{u(t,x)\}_{(t,x)\in\mathscr{D}}$ with covariance matrix function $C_u(t_1,x_1,t_2,x_2)$

$$u(t,x;\omega)=\sum_{n=1}^{\infty}\xi_n(\omega)\psi_n(t,x) \text{ in } L^2(\Omega,\mathbb{R}^n)$$

with

$$\begin{cases} \psi_n: & \text{eigenfunctions } (Q\psi)(t,x)=\int_{\mathscr{D}} C_X(t,x,t',x')\psi(t',x')dt'dx' \\ & \int_{\mathscr{D}} <\psi_n(t,x)\psi_m(t,x)>dtdx=\begin{cases} 1 & \text{if } n=m \\ 0 & \text{if } n\neq m \end{cases} \text{ in } L^2(\mathscr{D},\mathbb{R}^n) \\ \xi_n: & \text{scalar random variables } \xi_n(\omega)=\int_{\mathscr{D}} <X(t,x,\omega),\psi_m(t,x)>dtd \\ & E(\xi_n\xi_m)=\begin{cases} \lambda_n & \text{if } n=m \\ 0 & \text{if } n\neq m \end{cases} \end{cases}$$

Approach 2

For fixed $t \in \mathscr{D}_t$,
Apply the KL theorem to random field $\{u(t,x)\}_{x\in\mathscr{D}_x}$
with the covariance matrix function $C_u(t,x_1,t,x_2)$

$$u(t,x;\omega) = \sum_{n=1}^{\infty} \xi_n(t,\omega)\psi_n(t,x) \text{ in } L^2(\Omega,\mathbb{R}^n)$$

with

$$\begin{cases} \psi_n(t,.): & \text{eigenfunctions } (Q\psi)(x) = \int_{\mathscr{D}_x} C_X(t,x,t,x')\psi(x')dx' \\ & \int_{\mathscr{D}_x} < \psi_n(t,x)\psi_m(t,x) > dtdx = \begin{cases} 1 & \text{if} \quad n=m \\ 0 & \text{if} \quad n\neq m \end{cases} \text{ in } L^2(\mathscr{D}, \\ \xi_n(t,.): & \text{scalar random variables } \xi_n(t,\omega) = \int_{\mathscr{D}_x} < X(t,x,\omega), \psi_m(t,x \\ & E(\xi_n(t,.)\xi_m(t,.)) = \begin{cases} \lambda_n & \text{if} \quad n=m \\ 0 & \text{if} \quad n\neq m \end{cases} \end{cases}$$

Approach 3

$L^2([0,T]\times\Omega,\mathbb{R}^n)$ with

$< Y,Z >_{[0,T]\times\Omega} = \mathscr{E}(< Y,Z >)$ with $\mathscr{E}(.) = \frac{1}{T}\int_0^T E(.)dt.$

Apply KL theorem the random field $\{u(.,x)\}_{x\in\mathscr{D}_x}$
with the covariance matrix function $\mathscr{C}_u(\mathbf{x},\mathbf{x}') = \mathscr{E}(u(.,x)\otimes u(.,x'))$

$$u(x;t,\omega) = \sum_{n=1}^{\infty}\xi_n(t,\omega)\psi_n(x)$$

with

$$\begin{cases} \psi_n(.): & \text{eigenfunctions } (Q\psi)(x) = \int_{\mathscr{D}_x}\mathscr{C}_u(x,x')\psi(x')dx' \\ & \int_{\mathscr{D}_x} < \psi_n(x)\psi_m(x) > dx = \begin{cases} 1 & \text{if} \quad n=m \\ 0 & \text{if} \quad n\neq m \end{cases} \text{ in } L^2(\mathscr{D}_x,\mathbb{R}^n) \\ \xi_n: & \text{scalar random processes } \xi_n(t,\omega) = \int_{\mathscr{D}_x} < u(t,x,\omega)\psi_m(x) > d \\ & \mathscr{E}(\xi_n\xi_m) = \begin{cases} \lambda_n & \text{if} \quad n=m \\ 0 & \text{if} \quad n\neq m \end{cases} \text{ in } L^2([0,T]\times\Omega,\mathbb{R}) \end{cases}$$

Approach 3: discrete case, random process: $\{u(t)\}_{[0,T]}$

$$\begin{array}{rcl} u\colon [0,T]\times\Omega & \to & \mathbb{R}^n \\ (t,\omega) & \mapsto & u(t;\omega) \end{array}$$

$L^2([0,T]\times\Omega,\mathbb{R}^n)$ with

$< Y,Z >_{[\,0,T]\times\Omega} = \mathscr{E}(< Y,Z >)$ with $\mathscr{E}(.) = \frac{1}{T}\int_0^T E(.)dt.$

Covariance matrix $\mathscr{C}_u = \mathscr{E}(u(.)u(.)^T)$

$$u(t,\omega) = \sum_{n=1}^{\infty} \xi_n(t,\omega)\psi_n$$

$$\begin{cases} \psi_n: & \text{eigenvectors } \mathscr{C}_u\psi_n = \lambda\psi_n \\ & \psi_n\psi_m^T = \begin{cases} 1 & \text{if} \quad n=m \\ 0 & \text{if} \quad n\neq m \end{cases} \text{ in } \mathbb{R}^n \\ \\ \xi_n: & \text{scalar random processes } \xi_n(t,\omega) =< u(t,\omega)\psi_m > \\ & \mathscr{E}(\xi_n\xi_m) = \begin{cases} \lambda_n & \text{if} \quad n=m \\ 0 & \text{if} \quad n\neq m \end{cases} \text{ in } L^2([0,T]\times\Omega,\mathbb{R}) \end{cases}$$

Some remarks

1. The existence of the Karhunen-Loève expansion as described in approach 3 does not require any assumption on stationarity and ergodicity properties.
2. The Karhunen-Loève expansion as described in approach 3 usually depends on the time parameter T.
3. If the random field $\{u(t,x)\}_{(t,x)\in\mathscr{D}_t\times\mathscr{D}_x}$ is weakly stationary with respect to the time variable, $(C_u(t,x,t',x') = C_u(t-t',x,x'))$ then approach 2 and approach 3 give the same results. Moreover, the KL expansion does not depends on the time parameter T.

KL modes and Equivalent linear system

Let us consider the discrete nonlinear case: $U(t) \in \mathbb{R}^n$

$$M\ddot{U}(t) + C\dot{U}(t) + F(U(t)) = Bw(t)$$

where $\{w(t)\}_t$ is a Gaussian white noise process

$\{U(t)\}$: stationary process

Then, the KL modes of the stationary nonlinear response coincide with the KL modes of the stationary response of the associated equivalent linear system given by the true stochastic linearization method:

$$M\ddot{U}(t) + C\dot{U}(t) + K_{eq}U(t) = Bw(t)$$

where K_{eq} minimizes $E((F(U) - KU)^T(F(U) - KU))$

How to compute the PO modes?

The techniques to estimate the covariance function of a random field depends on its properties. If the random field is non-stationary, there is a general method for estimating its second-moment characteristics that assumes several realizations of the random field are available.
If the random field is weakly stationary with respect to the time variable, there is a method for estimating $\mathscr{C}_u(x_1, x_2)$ based on a single realization of the random field.
Two methods can be used to compute de PO modes:
- Direct method
- Snapshot method

Example

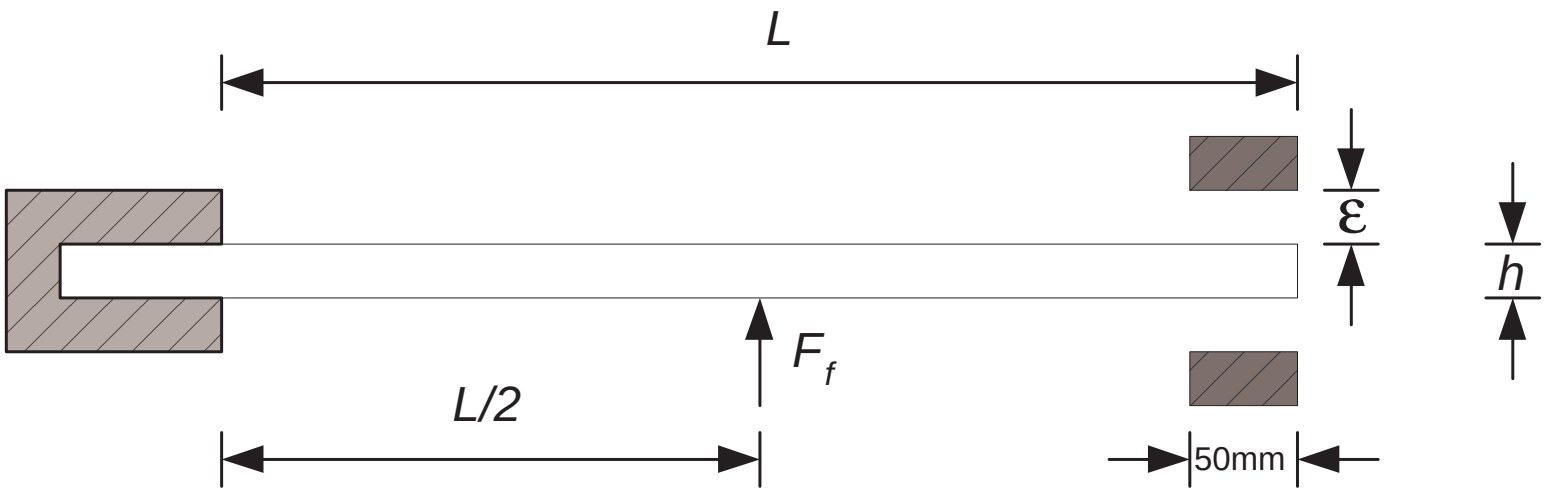


Figure: Vibro-impact beam

Some results are presented here obtained from simulated data generated from a mathematical model of a linear clamped beam impacting a flexible barrier.

Simulation of the experiment

$$EI\frac{\partial^4 w(x,t)}{\partial x^4}+\rho A\frac{\partial^2 w(x,t)}{\partial t^2}=F_f(t)\delta\left(x-x_f\right)+\sum_{i=1}^{N}F_b(w(x_{ci},t))\delta\left(x-x_{ci}\right)$$

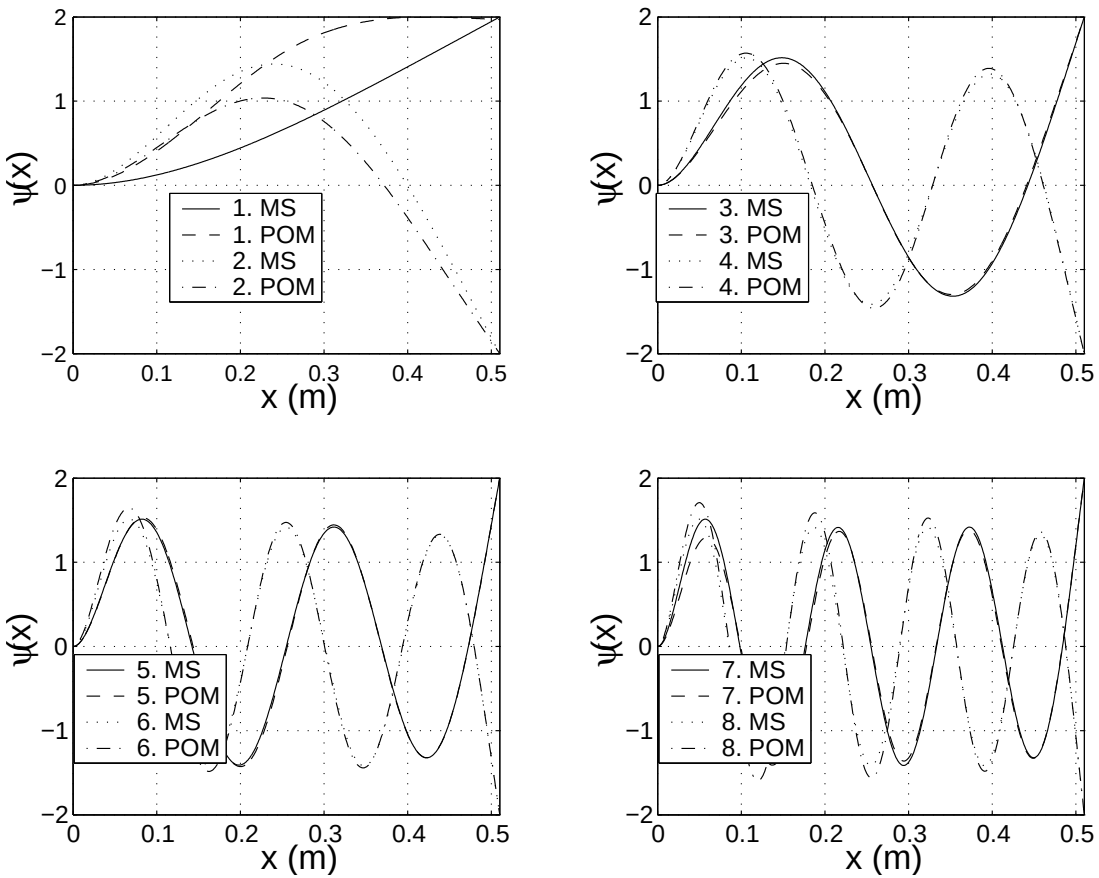
- Ten mode shapes of the associated linear system

$$\hat{w}(x,t)=\sum_{i=1}^{10}q_i(t)\phi_i(x)$$

- Galerkin method (10 DOF)

$$\ddot{Q}+[2\omega_i\tau_i]\dot{Q}+[\omega_i^2]Q+F_{ci}(Q)=BF_f(t)$$

where modal damping were added to the discretized model.



Comparison between PO modes ψ_i and linear modes ϕ_i

The first two KLs significantly differs from the first two mode shapes reflecting the influence of the barrier upon the system.

Reduced-order model formulation:

From the PO modes ψ_i, we have construct a reduced model.

$$EI\frac{\partial^4 w(x,t)}{\partial x^4}+\rho A\frac{\partial^2 w(x,t)}{\partial t^2}=F_f(t)\delta\left(x-x_f\right)+\sum_{i=1}^{N}F_b(w(x_{ci},t))\delta\left(x-x_{ci}\right)$$

- n KL modes with $(1 \leq n \leq 10)$

$$\hat{w}(x,t)=\sum_{i=1}^{n}a_i(t)\psi_i(x)$$

- Galerkin method (n DOF)

$$\ddot{A}+[2\omega_i\tau_i]\dot{A}+F_{KL}(A)=B_{KL}F_f(t)$$

where same modal damping were added to the discretized model.

This figure presents a comparison between the original and the reduced-order models constructed with 5 and 10 PO modes.

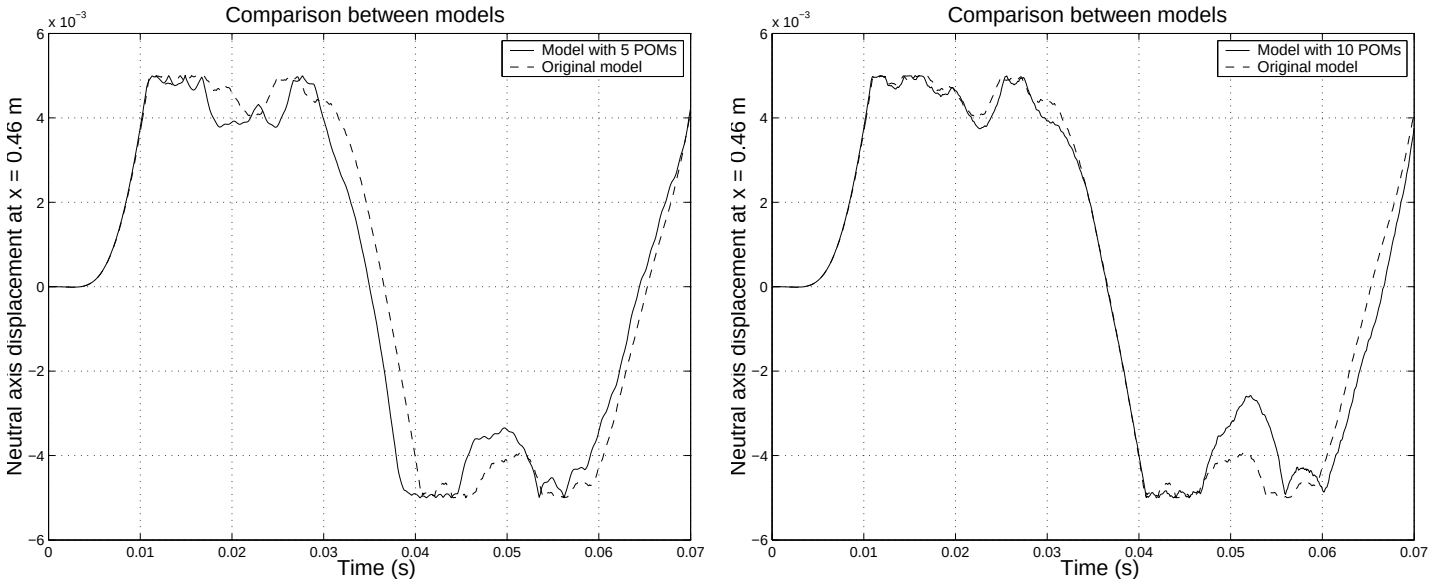


Figure: Dynamic with reduced order modes with 5 and 10 PO modes

The result is clearly not as good as expected and the full reduced-order model is not yet capable of reproducing the original response.
A probable explanation for this result is that the use of the modal damping ratios for the first and second KLMs is inappropriate as they are physically different.

Concluding remarks

- In order to use the KL theory to expand a random field in the variables separated (time-random variables and spatial variable), it is necessary to use the adequate spatial covariance function.
- The use of the PO modes to develop the reduced-order model in the presence of damping may not be robust.